Απειροστικός Λογισμός ΙΙ — Μάθημα 14 — 07/04/16 — ①

$$\blacktriangleright \int_{-2}^{2} \frac{1}{x^2-4}\,dx = \int_{-2}^{2} \frac{1}{(x+4)(x-4)}\,dx$$

Ανάλυση σε απλά κλάσματα:

$$\frac{1}{(x+4)(x-4)} = \frac{A}{(x+4)} + \frac{B}{(x-4)} = \frac{A(x-4)+B(x+4)}{(x+4)(x-4)} = \frac{(A+B)x+4B-4A}{(x+4)(x-4)}$$

Άρα

$$\left.\begin{matrix} A+B=0 \\ 4B-4A=1 \end{matrix}\right\} \Leftrightarrow \left.\begin{matrix} A=-B \\ 4B+4B=1 \end{matrix}\right\} \Leftrightarrow \left.\begin{matrix} A=-B \\ B=1/8 \end{matrix}\right\} \Leftrightarrow \left.\begin{matrix} A=-1/8 \\ B=1/8 \end{matrix}\right\}$$

Έτσι,

$$I = \left(-\frac{1}{8}\right)\int_{-2}^{2}\frac{1}{x+4}\,dx + \frac{1}{8}\int_{-2}^{2}\frac{1}{x-4}\,dx =$$

$$= -\frac{1}{8}\Big[\log(x+4)\Big]_{-2}^{2} + \frac{1}{8}\Big[\log(|x-4|)\Big]_{-2}^{2} =$$

$$= -\frac{1}{8}\Big[\log(x+4)\Big]_{-2}^{2} + \frac{1}{8}\Big[\log(4-x)\Big]_{-2}^{2} =$$

$$= -\frac{1}{8}(\log 6 - \log 2) + \frac{1}{8}(\log 2 - \log 6) =$$

$$= -\frac{1}{8}\log 3 - \frac{1}{8}\log 3 = -\frac{1}{4}\log 3.$$

▶ Παραδείγματα ανάλυσης σε απλά κλάσματα, χωρίς υπολογισμό συντελεστών

① $$\frac{2x^2+7}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{\Gamma}{(x-1)^3}$$

② $$\frac{P(x)}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx+\Gamma}{(x^2+1)}$$

$\deg P(x) \leq 2$

③ $$\frac{P(x)}{(x^2+4)^2(x-3)^3} = \frac{A_1}{(x-3)} + \frac{A_2}{(x-3)^2} + \frac{A_3}{(x-3)^3} + \frac{B_1x+\Gamma_1}{(x^2+4)} + \frac{B_2x+\Gamma_2}{(x^2+4)^2}$$

$\deg P(x) \leq 6$

Παραδείγματα ανάλυσης σε απλά κλάσματα με υπολογισμό:

① $I = \displaystyle\int \frac{x^3+2x^2+3x+1}{x^2-3x+2}\,dx$

Ο βαθμός του αριθμητή μεγαλύτερος του βαθμού του παρονομαστή. Συνεπώς, κάνουμε την ευκλείδεια διαίρεση.

$$
\begin{array}{r|l}
x^3+2x^2+3x+1 & x^2-3x+2 \\ \cline{2-2}
-x^3+3x^2-2x & \underbrace{x+5}_{\text{πηλίκο}} \\ \cline{1-1}
5x^2+x+1 & \\
-5x^2+15x-10 & \\ \cline{1-1}
\underbrace{16x-9}_{\text{υπόλοιπο}} &
\end{array}
$$

Άρα, από την ταυτότητα της ευκλείδειας διαίρεσης προκύπτει:

$$x^3+2x^2+3x+1 = (x^2-3x+2)(x+5) + (16x-9)$$

Έτσι, $I = \displaystyle\int x+5+\frac{16x-9}{x^2-3x+2}\,dx = \int x+5\,dx + \int \frac{16x-9}{x^2-3x+2}\,dx$

$$\frac{16x-9}{x^2-3x+2} = \frac{16x-9}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-1)}{(x-2)(x-1)} =$$

$$= \frac{(A+B)x-2A-B}{(x-1)(x-2)}$$

$$\left.\begin{array}{l} 16x = (A+B)x \\ -9 = -2A-B \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} A+B=16 \\ 2A+B=9 \end{array}\right\} \Rightarrow \left.\begin{array}{l} A+B=16 \\ A=9-16 \end{array}\right\} \Rightarrow \left.\begin{array}{l} B=23 \\ A=-7 \end{array}\right\}.$$

Άρα, $I = \displaystyle\int x+5+\frac{(-7)}{(x-1)}+\frac{23}{(x-2)}\,dx = \frac{x^2}{2}+5x-7\log(|x-1|)+23\log(|x-2|) + c.$

② $I = \displaystyle\int \frac{x^3}{x^4+1}\,dx.$

α' τρόπος: $I = \displaystyle\int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\log(x^4+1)+c$

β' τρόπος: (Παραγοντοποίηση παρονομαστή - ανάλυση σε απλά κλάσματα)

$$x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}x)^2 =$$
$$= (x^2+\sqrt{2}\,x+1)(x^2-\sqrt{2}\,x+1)$$

$$\frac{x^3}{x^4+1} = \frac{x^3}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{\Gamma x+\Delta}{x^2-\sqrt{2}x+1} \quad \text{κ.τ.λ.}$$

3) $$I = \int_1^{\sqrt{3}} \frac{-3x-4}{x^3-2x^2+x-2}\,dx =$$

$\alpha\cdot\pi + \log(\beta+\gamma\cdot\sqrt{3})$
Το αποτέλεσμα να εκφραστεί σε αυτή τη μορφή, με $\alpha,\beta,\gamma \in \mathbb{Q}$

$$= \int_1^{\sqrt{3}} \frac{-3x-4}{(x-2)(x^2+1)}\,dx$$

$$\frac{-3x-4}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{Bx+\Gamma}{(x^2+1)} = \frac{A(x^2+1)+(Bx+\Gamma)(x-2)}{(x-2)(x^2+1)} =$$

$$= \frac{Ax^2+A+Bx^2+\Gamma x-2Bx-2\Gamma}{(x-2)(x^2+1)} = \frac{(A+B)x^2+(\Gamma-2B)x+A-2\Gamma}{(x-2)\cdot(x^2+1)}$$

Άρα, προκύπτει:
$$\left.\begin{aligned} A+B&=0\\ \Gamma-2B&=-3\\ A-2\Gamma&=-4\end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} A&=-B\\ \Gamma+2A&=-3\\ A-2\Gamma&=-4\end{aligned}\right\} \Leftrightarrow$$

$$\Leftrightarrow \left.\begin{aligned} A&=-B\\ 2\Gamma+4A&=-6\\ 5A&=-10\end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} A&=-2\\ B&=2\\ \Gamma&=1\end{aligned}\right\}$$

$$I = \int_1^{\sqrt{3}} \frac{-2}{(x-2)} + \frac{2x+1}{(x^2+1)}\,dx =$$

$$= \int_1^{\sqrt{3}} \frac{-2}{(x-2)} + \frac{2x}{(x^2+1)} + \frac{1}{(x^2+1)}\,dx = -2\Big[\log(|x-2|)\Big]_1^{\sqrt{3}} + \Big[\log(|x^2+1|)\Big]_1^{\sqrt{3}} + \Big[\arctan x\Big]_1^{\sqrt{3}}$$

$$= -2\left(\log(2-\sqrt{3}) - \cancel{\log(2-1)}^{0}\right) + \log 4 - \log 2 + \arctan\sqrt{3} - \arctan 1 =$$

$$= -2\log(2-\sqrt{3}) + \log 2 + \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} + \log\frac{2}{(2-\sqrt{3})^2} =$$

$$= \frac{\pi}{12} + \log\left(\frac{2}{4-4\sqrt{3}+3}\right) = \frac{\pi}{12} + \log\frac{2}{7-4\sqrt{3}} =$$

$$= \frac{\pi}{12} + \log\frac{2\cdot(7+4\sqrt{3})}{7^2-(4\sqrt{3})^2} = \frac{\pi}{12} + \log(14+8\sqrt{3})$$

④ $\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\left(\frac{x^2}{4}+1\right)}\,dx =$ ④

$= \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx = 2\cdot\frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\cdot\left(\frac{x}{2}\right)'\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+c.$

⑤ $\int \frac{2x+7}{x^2+4x+5}\,dx = \int \frac{2x+4}{x^2+4x+5} + \frac{3}{x^2+4x+5}\,dx =$

$= \int \frac{2x+4}{x^2+4x+5}\,dx + 3\int \frac{1}{x^2+4x+5}\,dx =$

$= \log(|x^2+4x+5|) + 3\int \frac{1}{(x+2)^2+1}\,dx =$

$= \log(|x^2+4x+5|) + 3\arctan(x+2) + c.$